TOPICS IN COMPLEX ANALYSIS @ EPFL, FALL 2024
HOMEWORK 11

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Homework 11.1 (Cauchy–Riemann equations and **C**-linearity on $\mathbf{C}^n$)**.** Identify $\mathbf{C}^n$ with $\mathbf{R}^{2n}$ and let $U \subset \mathbf{C}^n$. Consider a differentiable function $f: U \to \mathbf{C}$. Then at each point $a \in U$ there exists an **R**-linear mapping $\mathrm{D}f(a): \mathbf{R}^{2n} \to \mathbf{C}$ such that

$$\lim_{\substack{h \to 0, \\ h \neq 0}} \frac{|f(a+h) - f(a) - Df(a)h|}{|h|} = 0.$$

Show $\mathrm{D}f(a)$ is **C**-linear if and only if

$$\frac{\partial}{\partial \bar{z}_j} f(a) = 0$$

for every $j \in \{1, \ldots, n\}$, where $2\partial/\partial\bar{z}_j := \partial/\partial x_j + \mathrm{i}\,\partial/\partial y_j$ and $z = x + \mathrm{i}y$ with $x, y \in \mathbf{R}^n$.

Homework 11.2 (Slicing method in action)**.** In this exercise we transfer some well-known results from one-dimensional complex analysis to the several variables setting. Show the following statements.

a. **Liouville's theorem**. Every bounded entire function $f: \mathbf{C}^n \to \mathbf{C}$ is constant.
b. **Identity theorem**. Let $D \subset \mathbf{C}^n$ be a domain and $f: D \to \mathbf{C}$ be holomorphic. If f vanishes identically on $B_r(a)$ for some $a \in D$ and $r > 0$, then $f = 0$.
c. **Open mapping theorem**. Let $D \subset \mathbf{C}^n$ be a domain and $f: D \to \mathbf{C}$ be nonconstant and holomorphic. Then $f(D)$ is again a domain.
d. **Maximum principle**. Let $D \subset \mathbf{C}^n$ be a domain and $f: D \to \mathbf{C}$ be holomorphic. If $|f|$ attains its maximum on D then f is constant.

Homework 11.3 (Failure of the open mapping theorem in the fully vectorial case)**.** In Homework 11.2 we proved the open mapping theorem for functions with target domain **C**. Here we show that it is false for vectorial functions $f: D \to \mathbf{C}^m$, where $m \geq 2$, even when no component is constant. Define $f: \mathbf{C}^2 \to \mathbf{C}^2$ by $f(z_1, z_2) := (z_1, z_1 z_2)$. Show f is holomorphic yet not an open map[1].

Homework 11.4 (Power series in several variables*)**.** a. For each series below, determine for each series below the largest open set $U \subset \mathbf{C}^2$ where it converges absolutely. Is it convex?

- $\displaystyle\sum_{n=0}^{\infty} z^n w^n$.
- $\displaystyle\sum_{n=1}^{\infty} z^n w^{n!}$.

b. Let $F(z) := \sum_{\alpha \in \mathbf{N}_0^n} c_\alpha z^\alpha$ be a formal power series centered at the origin. Show that if $z \in \mathbf{C}^n$ is such that $F(z)$ converges absolutely, then $F(\lambda_1 z_1, \ldots, \lambda_n z_n)$ also converges absolutely provided $|\lambda_i| \leq 1$ for every $i \in \{1, \ldots, n\}$.

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Date: December 9, 2024.

[1]**Hint.** In order to guess where the map is not open one can look where its differential is not invertible.